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10760. A function $f : \mathbb{N} \to \mathbb{N}$ is completely multiplicative is $f(1) = 1$ and $f(mn) = f(m)f(n)$ for all positive integers m and n. Find all completely multiplicative functions f with the property that the function $F(n) = \sum_{k=1}^{n} f(k)$ is also completely multiplicative.

Solution. Let f be a completely multiplicative function such that F as defined above is also completely multiplicative. Let $y = f(2)$. Given an odd prime p, let us assume that for all $1 < n < p$, $f(n) = y^{a_n}$, where a_n is an integer, an hypothesis which is clearly satisfied for $p = 3$. Since $p + 1 = 2m$ for some integer m and $F(p+1) = F(p-1) + f(p) + f(p+1)$, we have

$$\begin{aligned} f(p) &= F(p+1) - F(p-1) - f(p+1) \\ &= F(2)F(m) - F(2)F(m-1) - f(2)f(m) \\ &= (1+y)(F(m) - F(m-1)) - f(2)f(m) \\ &= (1+y)f(m) - yf(m) \\ &= f(m) \\ &= y^{a_m}. \end{aligned} \tag{1}$$

Hence by induction on all primes p and using the multiplicative property of f, we have shown that $f(n) = y^{a_n}\ \forall\ n \geq 2$. Using (1), we find that $f(3) = y$, $f(5) = y$ and $f(7) = y^2$. With these values, one can verify that the equation $F(10) = F(2)F(5)$ reduces to $y^2 = y$, hence $y = 0$ or $y = 1$.

In conclusion, there are only two completely multiplicative functions f which satisfy the given conditions, and they are $f(n) = 0\ \forall\ n > 1$, in which case $F(n) = 1$, and $f(n) = 1$, in which case $F(n) = n$.

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