Problems in *The American Mathematical Monthly*

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February 2000

10774. Let $F(1) = F(2) = 1$ and $F(n) = F(n-1) + F(n-2)$ for $n \geq 3$. Show that

$$\big(F(F(1998))\big)^2 + \big(F(F(1999))\big)^2 = F(F(1997))F(F(2000)). \tag{1}$$

Solution. Let us define $F(0) = 0$. By induction on k, one can easily show that $\forall n \in N$

$$\begin{aligned} F(n+k) &= F(k)F(n+1) + F(k-1)F(n) \\ &= F(k)F(n+1) + \big(F(k+1) - F(k)\big)F(n). \end{aligned}$$

Using this relation, we then compute, for $m, b \in \mathbb{N}$,

$$\begin{aligned} F(2m+1) &= F(m + (m+1)) \\ &= F(m)^2 + F(m+1)^2 && (2) \\ F(2m+b+1) &= F(2m + (b+1)) \\ &= F(b+1)F(2m+1) + F(b)F(2m) \\ &= F(b+1)\big[F(m)^2 + F(m+1)^2\big] \\ &\quad + F(b)\big[F(m)F(m+1) + \big(F(m+1) - F(m)\big)F(m)\big] && (3) \\ F(2m+2b+1) &= F(2(m+b)+1) \\ &= F(m+b)^2 + F(m+(b+1))^2 \\ &= \big[F(b)F(m+1) + \big(F(b+1) - F(b)\big)F(m)\big]^2 \\ &\quad + \big[F(b+1)F(m+1) + F(b)F(m)\big]^2. && (4) \end{aligned}$$

The equalities (2), (3) and (4) are then used to compute

$$\begin{aligned} &F(b)^2 + F(2m+b+1)^2 - F(2m+1)F(2m+2b+1) = \\ &\quad -F(b)^2\big[F(m)^2 + F(m)F(m+1) - F(m+1)^2 + 1\big] \\ &\qquad \big[F(m)^2 + F(m)F(m+1) - F(m+1)^2 - 1\big], \end{aligned}$$

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an expression which is zero since by induction on k one can easily show that

$$F(k)^2 + F(k)F(k+1) - F(k+1)^2 = \pm 1.$$

Writing $a = 2m+1$, we have shown that for $a, b \in \mathbb{N}$, a odd,

$$F(b)^2 + F(a+b)^2 = F(a)F(2b+a).$$

Equation (1) is the particular case $a = F(1997)$ and $b = F(1998)$. $F(1997)$ is an odd integer since $1997 \equiv 2 \bmod 3$ and one can easily show that the parity of $F(k+3)$ is the same as that of $F(k)$ for all $k \in \mathbb{N}$.